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OSCILLATORY PROPERTIES OF THE SOLUTIONS OF STANDARD LINEAR SETS OF DIFFERENTIAL EQUATIONS

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We shall examine the system /533†

$$dx/dt = JH(t)x, \tag{1}$$

where x is a 2k-dimensional vector; $H(t) = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}$; $\alpha = \alpha^*$, $\gamma = \gamma^*$; $J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$; I_k is a unit matrix of the order k; $H(t) = H(t)^*$ is a real 2k x 2k symmetrical matrix function, $0 \leq t < \infty$.

We shall assume that all coefficients of the differential equations, unless otherwise stipulated, are real functions, integrable with Lebesque integrals at any finite interval.

1. We will designate as $\boxed{\mathfrak{S}}$ the group of simplectic matrices, that is real 2k x 2k matrices X, satisfying the relation $X^* JX = J$. As is well known, for any t the matrizant of equation (1) $X(t) \in \boxed{\mathfrak{S}}$.

I.M. Gel'fand and V.B. Lipskiy showed (1) that the group $\boxed{\mathfrak{S}}$ is homeomorphic to the topological space on the circle $\boxed{\mathfrak{K}}$. The angle ϕ, determining the position of a point on $\boxed{\mathfrak{K}}$, is called (1) the argument of the corresponding matrix X, $\phi = \mathrm{Arg}_0 X$*. $\mathrm{Arg}_0 X$ is a

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†Numbers in right-hand margin indicate pagination in the foreign text.

*We shall present this definition in greater detail. Let X = SU, where S is a positively defined, and U an orthogonal, matrix. From $X \in \boxed{\mathfrak{S}}$ it follows (1) that $S \in \boxed{\mathfrak{S}}$, $U \in \boxed{\mathfrak{U}}$. The matrix U has the form $U = \begin{pmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{pmatrix}$, where $w = u_1 + iu_2$ is a unitary matrix, $\det w = e^{i\rho}$. By definition $\phi = \mathrm{Arg}_0 X$.

multiple-valued function on $\boxed{\mathfrak{G}}$, each of the branches of which is continuous; therefore, the increment of the argument along the continuous curve $X(t)$, designated as $\Delta \mathrm{Arg}_0 X(t) = 2n\pi$ is defined uniquely. The integer n is call the index of the closed curve $X(t)$.

Definition 1. We shall call the argument of the simplectic matrix X any calculating real function determined on $\boxed{\mathfrak{G}}$ so that: 1) if $(\mathrm{Arg} X_0)$ is one of the values of Arg X, then the remaining values are $(\mathrm{Arg}\ X)_n = (\mathrm{Arg}\ X)_0 + 2n\pi$, $n = \pm 1, \pm 2, \ldots$; 2) each of the values of $(\mathrm{Arg}\ X)_n$ is a continuous function X; 3) there exists a continuous function X so that $\mathrm{Arg} U_0(t) = 2\pi$*. The following theorem shows that all similar "arguments" are, in a certain sense, equally justified.

Theorem 1. If $X(t)$ is a closed curve of index n, then /534
$\Delta \mathrm{Arg}\ X(t) = 2n\pi$. The closed curve $X(t)$ may be subtended in the group $\boxed{\mathfrak{G}}$ in a point then and only then if $\Delta \mathrm{Arg} U_0(t) = 0$.

Let $X = \begin{pmatrix} U_1 & U_2 \\ V_1 & V_2 \end{pmatrix} \in \boxed{\mathfrak{G}}$. We shall cite some of the possible definitions of the argument of the matrix X:

$\mathrm{Arg}_1 X = \mathrm{Arg}\ \det(U_1 - iU_2)$, $\mathrm{Arg}_2 X = \mathrm{Arg}\ \det\ (V_1 - iV_2)$, $\mathrm{Arg}_3 X = \mathrm{Arg}\ \det(U_1 - iV_1)$, $\mathrm{Arg}_4 X = \mathrm{Arg}\ \det(U_2 - iV_2)$, $\mathrm{Arg}_5 X = \sum_{j=1}^{k} \mathrm{Arg} \rho_j^{(+)}$,

*Property 2) is understood in the following sense: for any matrix $X_0 \in \boxed{\mathfrak{G}}$ and any fixed value of Arg X_0, which we shall designate $(\mathrm{Arg}\ X_0)_n$, for any $\varepsilon > 0$ it is possible to indicate $\delta > 0$ so that where $X - X_0 < \delta$ a value of Arg X will be found, which we shall designate as $(\mathrm{Arg}\ X)_n$, such that $|(\mathrm{Arg}\ X_0)_n - (\mathrm{Arg}\ X)_n| < \varepsilon$. In this case, for any continuous curve $S(t)$ the values $(\mathrm{Arg}\ X(t))_n$, $n = 0, \pm 1, \pm 2, \ldots$, form the calculating number of the continuous functions. Therefore, the increment $\mathrm{Arg}\ X(t) = (\mathrm{Arg}\ X(1))_n - (\mathrm{Arg}\ X(0))_n$ is uniquely defined, which justifies the formulation of property 3).

where $\rho_j^{(+)}$ are eigenvalues of the first type of the matrix X*. The definition Arg X was used in reference (3). The proof of theorem 1 repeats the proof of theorem 1.3 in reference (3).

The arguments Arg'X and Arg"X will be called equivalent if there exists a constant $c>0$ so that, for any continuous curve $X(t) \in |\mathfrak{S}|$, $|\Delta Arg'X(t) - \Delta Arg''X(t)| < c$ is fulfilled.

The following theorem plays an important role for the discussion given below:

Theorem 2. All arguments $Arg_j X$, $j = 0, \ldots, 5$ are equivalent.

It is possible to cite examples of non-equivalent definitions of the argument. Below we shall assume Arg X to be one of $Arg_j X$ or some definition of the argument equivalent to that enumerated.

2. Let X(t) be the matrizant of equation (1).

Definition 2. Equation (1) is called oscillatory if Arg X(t) is unbounded and non-oscillatory if Arg X(t) is a bounded function where $t \to +\infty$.

We shall call equation (1) an equation of the positive type if there exists a definition of the argument $Arg\,X = Arg_* X$, equivalent to $Arg_j X$, $j = 0 \ldots, 5$ so that $Arg_* X(t)$ is a monotonically increasing function t. Thus, in the case of the oscillatory nature of equation (1) of the positive type the matrizant X(t) where $t \to \infty$ is infinitely "twisted" in the "torus" $|\mathfrak{S}|$ and in

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*In addition to M.G. Kreyn's (2) definition of the type of eigenvalues lying on a unit circle, we here consider the eigenvalues lying inside a unit circle to be eigenvalues of the first type and those outside it to be of the second type.

the case of nonoscillation a similar "angle of twist"

$\Delta \mathrm{Arg}\ X(t)\Big|_{t=0}^{t=\infty}$ is an infinite value.

Let $U(t)$, $V(t)$ be an absolutely continuous matrix-function satisfying the conditions: a) $UV_* = VU^*$; b) from $Uz = Vz = 0$ it follows that $z = 0$. Let $R = UU^* + VV^*$, and $G = \frac{dU}{dt}V^* - \frac{dV}{dt}U^*$. Then $G = G^*$, $R = R^*>0$ and $\frac{d}{dt}$ Arg det $(U + iV) = \sum_{j=1}^{k} \lambda_j(t)$, where $\lambda_j(t)$ are the roots of the equation det $[G(t) - \lambda R(t)] = 0$. Using this formula or some complementing the proof of theorem 2 and 4 (4), we obtain:

Theorem 3. Let in equation (1) $\gamma(t) \geq 0$ and $S_p\gamma(t) > 0$. Then equation (1) is an equation of the positive type and $\mathrm{Arg}_* X = \mathrm{Arg}_1 X$.

The theorem remains valid if the matrix α is taken instead of the matrix γ (then $\mathrm{Arg}_* X = \mathrm{Arg}_2 X$).

We shall call the system of vectors $x_1, \ldots, x_k$ an isotropic system if $(Jx_j, x_h) = 0$, $j, h = 1, \ldots, k$. Since for any solutions $x_1(t)$, $x_2(t)$ of equation (1) $(Jx_1(t), x_2(t)) =$ const, then, in order to obtain an isotropic system of equations, it is sufficient to take any isotropic system of vectors as initial values. We shall designate as $||x_1, \ldots, s_2|| = \binom{U}{V}$ the matrix with the columns $x_1, \ldots, x_k$ (U and V are square k x k matrices).

We shall call the scalar function $\zeta(t)\not\equiv 0$ oscillatory if it has infinitely many zeroes $t_n \to +\infty$ and nonoscillatory in the opposite case.

Theorem 4. Let in equation (1) $\gamma(t) \geq 0$* and $x_1, \ldots, x_k$ be /535

*Here and below the condition $\gamma \geq 0$ $(\gamma > 0)$ for the summetrical matrix γ signifies that $(\gamma c, c) \geq 0$ $((\gamma c, c) > 0$ where $c \neq 0)$.

an arbitrary isotropic system of linearly independent real solutions, $||x, \dots |x_k|| = \binom{U}{V}$ so that $\det U(t) \neq 0$. Then equation (1) is oscillatory or non-oscillatory simultaneously with the function $\det U(t)$.

3. Let us consider the vector equation

$$\frac{d}{dt}\left[R(t)\frac{dy}{dt}\right] + P(t)y = 0, \tag{2}$$

where y is a k-dimensional function; $R(t) = R(t)^* > 0$; $P(t) = P(t)^*$; $R(t)^{-1}$ and $P(t)$ are real $k \times k$-matrix functions integrable with Lebesque integers at any finite interval. Assuming $x_1 = y$, $x_2 = R\frac{dy}{dt}$ and $x = \binom{x_1}{x_2}$, we reduce (2) to the equivalent equation (1) with the matrix $H(t) = \begin{pmatrix} P & 0 \\ 0 & R^{-1} \end{pmatrix}$ which we shall call corresponding equation (2). According to theorem 3 this equation will be an equation of the positive type.

Definition 3 (Sternberg, Bliss, Reid [5,6]). Equation (2) is called oscillatory if for any $t_2 < t_1 < t_0$ and a solution $y(t) \neq 0$ of equation (2) so that $y(t_1) = y(t_2) = 0$, and non-oscillatory in the opposite case.

Sternberg established [5] (on the basis of Bliss and Reid's results [6]) for equation (2) a theorem, close in formulation to theorem 4, where, however, oscillation is understood in the sense of definition 3. Using this theorem we obtain:

Theorem 5. Equation (2) is oscillatory (non-oscillatory) in the sense of definition 3 if and only if equation (1) corresponding to it is oscillatory (non-oscillatory) in the sense of definition 2.*

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*After establishing the equivalence of definitions 2 and 3 it is seen that theorem 4 somewhat strengthens Sternberg's corresponding assertion (lemma 3.1 [5] or lemma 2.1 [7]).

4. The scalar equation

$$\sum_{j=0}^{k}(-L)^{j}[\phi_{j}(t)\eta^{(k-j)}]^{(k-j)} = 0, \ \phi_{0}(t) > 0 \tag{3}$$

with the introduction of the vector x with the elements $\xi_j = \eta^{(j-1)}$, $j = 1, \ldots, k$, $\xi_{j+k} = \phi_{k-j}\xi_{j+1} - \xi_{k+j+1}$, $j = 1, \ldots, k$, $\xi_{2k} = \phi_0\xi_k$ reduces to equation (1) which we shall call corresponding equation (3). In this equation α is a diagonal matrix with the diagonal elements $-\phi_1, \ldots, -\phi_k$; γ is a diagonal matrix with the diagonal elements $0, \ldots, 0, \gamma_0^{-1}$ and β is a Jordan block corresponding to a zero eigenvalue.

Definition 4. (I. M. Glazman [8], L. D. Nikolenko [9]). Equation (3) is called oscillatory if for any $t_0 > 0$ there is found a solution $y(t) \not\equiv 0$ having at least two k-dimensional zeroes $t_2 > t_1 > t_0$ and non-oscillatory in the opposite case.

Theorem 6. Equation 3 is oscillatory (non-oscillatory) in the sense of definition 4 if and only if the equation (1) corresponding to it is oscillatory (non-oscillatory) in the sense of definition 2.

Using evaluation for different definitions of the argument it is possible to obtain different satisfactory conditions of the oscillation and non-oscillation of equations (1), (2), and (3); we will not deal with this here.

5. Let us examine the equation

$$dx/dt = J[H_0(t) + \lambda H_1(t)]_x \tag{4}$$

with self-conjugate boundary conditions, for example, of the form /536

$$x(\tau) = Sx(0), \tag{5}$$

where $S \in \mathfrak{S}$. We shall assume that almost everywhere in $(0, \tau)$ $H_1 \geq 0$ and that from the conditions $dz/dt = H_0(t)z$ and $H_1(t)z = 0$

it follows that $z = 0$. Given these assumptions, the eigenvalues λ_j may only be real; to the different eigenvalues there correspond the orthogonal latent vectors: $\int_0^{\tau}(H_1 x_j,\ x_h)dt = 0$ where $j \neq h$.

The classical case when $H_1(t) > 0$ almost everywhere is well studied. Difficulties occur only in the case of degeneration when $\det H_1(t) = 0$ on a set of positive degree. In this case there may exist a finite number of eigenvalues or they may not exist at all. In the works of M. G. Kreyn [2,10], together with other results, there is indicated the necessary and sufficient condition that boundary problem (4)-(5) (where $H_0 \equiv 0$) have at least one eigenvalue and also the sufficient condition that there be infinitely many of them. The following theorem may be useful in studying similar problems:

Theorem 7. We designate the matrizant of equation (4) as $X(t, \lambda)$. In order for the boundary problem (4)-(5) to have infinitely many eigenvalues $\lambda_n \to \infty$ ($\lambda_n \to -\infty$), it is necessary and sufficient that $\lim\limits_{\lambda\to\infty} \operatorname{Arg} X(\tau, \lambda) = \infty$ ($\lim\limits_{\lambda\to-\infty} \operatorname{Arg} X(\tau, \lambda) = -\infty$).

We note that under the assumptions which have been made, $\operatorname{Arg}_3 X(\tau, \lambda)$ is a non-decreasing function of λ; therefore, there exist in finite or finite limits $\lim\limits_{\lambda\to\pm\infty} \operatorname{Arg}_3 X(\tau, \lambda)$.

The geometrical sense of theorem 7 is obvious:
if $\lim\limits_{\lambda\to\infty} \operatorname{Arg} X(\tau, \lambda) = \infty$, then the "point" $X(\tau, \lambda)$, infinitely "twisted" in the "torus" $\mathfrak{G}$, hits the "surface" $\det(X-S) = 0$ an uncountable number of times; the corresponding values of λ will also be eigenvalues.

Let $h_1(t) \leq \ldots \leq h_{2k}(t)$ be eigenvalues of the matrix $H_1(t)$. It is possible to obtain the following criterion: if $\int_0^t \sum_{j=1}^{k} h_j(t)dt > 0$, then the boundary problem (4)-(5) has an uncountable number of eigenvalues λ_n, $n = 0, \pm 1, \pm 2, \ldots$, $\lambda_n \to \infty$, $\lambda_{-n} \to -\infty$ where $n \to \infty$.

From theorem 7 it also follows that the boundary problem for the equation $d^2y/dt^2 + [P(t) + \lambda Q(t)]y = 0$, $P(t) = P(t)^*$, $Q(t) = Q(t)^*$ almost everywhere, has an uncountable number of eigenvalues $\lambda_n \to \infty$ if $S_p Q(t) > 0$ on a set of positive degree.

In an analogous fashion I. M. Glazman's results [8] on the connection between the oscillation properties of equation of type (4) and the properties of the spectrum of the corresponding differential equations of the operators at an infinite interval $\tau=\infty$, acquire an obvious geometrical sense.

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